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ORDER PRESERVING NONEXPANSIVE OPERATORS IN L_1

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ABSTRACT

We study the limit behaviour of $T^k f$ and of Cesaro averages $A_n f$ of this sequence, when T is order preserving and nonexpansive in L_1^+. If T contracts also the L_∞-norm, the sequence $T^n f$ converges in distribution, and $A_n f$ converges weakly in L_p $(1 < p < \infty)$, and also in L_1 if the measure is finite. "Speed limit" operators are introduced to show that strong convergence of $A_n f$ need not hold. The concept of convergence in distribution is extended to infinite measure spaces.

1. Introduction

There is now a considerable literature on Markovian operators in L_1, i.e., on linear, order preserving operators T in L_1 of a measure space (Ω, Σ, μ), which preserve integrals: $\int Tf d\mu = \int f d\mu$ for all $f \in L_1$. These operators describe the random movement of matter. A nonnegative function f may be interpreted as a mass distribution in Ω. $\int_B f d\mu$ is the mass contained in B. Tf is the mass distribution after one time unit, if the mass is mapped by a stochastic kernel, see [Kr 1].

The linearity of T implies two properties of this model of "diffusion" of matter which may not always be realistic:

(1) The movement of the mass in a set B is not affected by the mass distribution in the complement B^c of B.

(2) Mass sitting in the top of the distribution f is mapped in exactly the same way as mass below it.

We therefore propose to study operators T in L_1^+ or in L_1 which are order

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preserving and integral preserving, but which need not be linear. Recall that T is called *nonexpansive* in L_p if $\|Tf - Tg\|_p \leqq \|f - g\|_p$ holds for all $f, g \in L_p$. Our operators are nonexpansive in L_1.

After some preliminaries we begin with a theorem asserting the convergence in distribution of $T^n f$ if T decreases the L_∞-norm. It seems that this simple but useful result has been overlooked until recently even in the linear case. After completing our work we learned that Akcoglu and Boivin [AB] obtained the same result (and more) for linear T in an as yet unpublished paper. We prove the result even in the case of σ-finite measure spaces. As we are not aware of any place where convergence in distribution is discussed for infinite measure spaces, we include such a discussion in this paper.

Our work has partly been motivated by the beautiful theorems of Baillon [Bai 1], [Bai 2], who showed that $A_n f = n^{-1} \Sigma_{k=0}^{n-1} T^k f$ converges weakly if T is nonexpansive in a Hilbert space or in L_p with $1 < p < \infty$. Our main result is a theorem on the weak convergence of $A_n f$ in L_p for $1 < p < \infty$, and for $p = 1$ in finite measure spaces. Here again, T as above is assumed also to decrease the L_∞-norm. Then, T decreases the L_p-norm, but it need not be nonexpansive in L_p for $1 < p < \infty$.

We then introduce a class of operators in L_1^+, called speed limit operators. They are nonexpansive in L_p^+ for $1 \leqq p \leqq \infty$, and they are used to show that $A_n f$ need not converge strongly. Usually, a rather intricate example of Genel and Lindenstrauss [GL] is utilized to show that strong convergence need not hold for Baillon's result in Hilbert space. For the present purpose, our example is both simpler and more informative.

2. Preliminaries

An operator T in L_p^+ or in L_p is called *norm decreasing* in L_p^+ (in L_p) if $\|Tf\|_p \leqq \|f\|_p$ holds for all f. T is called *order preserving* if $f \leqq g$ implies $Tf \leqq Tg$. Sometimes we use this terminology even if T is not defined on all of L_p or L_p^+.

LEMMA 2.1. *If T is order preserving in L_p^+ or in L_p and if $\|Tf - Tg\|_p \leqq \|f - g\|_p$ holds for all f, g with $f \geqq g$, then T is nonexpansive in L_p^+ (L_p).*

PROOF. For general f, g, we have $T(f \vee g) \geqq Tf$, $Tg \geqq T(f \wedge g)$. Hence $|Tf - Tg| \leqq T(f \vee g) - T(f \wedge g)$, and $\|Tf - Tg\|_p \leqq \|T(f \vee g) - T(f \wedge g)\|_p \leqq \|(f \vee g) - (f \wedge g)\|_p = \|f - g\|_p$.

LEMMA 2.2. *If T is order preserving in L_1^+ or L_1 and integral preserving, then T is nonexpansive in L_1^+ (resp. L_1).*

PROOF. For $f \leqq g$ we have $\| Tg - Tf \|_1 = \int (Tg - Tf) d\mu = \int (g - f) d\mu = \| g - f \|_1$.

Sometimes it is convenient to consider only the restrictions of an operator T in L_p to L_p^+ or to L_p^-. If T is order preserving in L_p and $T0 = 0$, then $T_{(+)}$ shall denote the restriction of T to L_p^+, and $T_{(-)}$ shall be the operator in L_p^+ defined by $T_{(-)}f = -T(-f)$. If T_1, T_2 are order preserving in L_p^+ and $T_1 0 = T_2 0 = 0$, then $(T_1 \ominus T_2)f = T_1 f^+ - T_2 f^-$ is an order preserving operator in L_p. If T_1 and T_2 are integral preserving, T is integral preserving. As T need not be linear, T can be different from $T_{(+)} \ominus T_{(-)}$. Yet, we shall see later that some results about T are possible by studying $T_{(+)}$ and $T_{(-)}$.

LEMMA 2.3. *If $T = T_{(+)} \ominus T_{(-)}$ is order preserving in L_1 and $T0 = 0$, and if $T_{(+)}$ and $T_{(-)}$ are nonexpansive in L_1^+, then T is nonexpansive in L_1.*

PROOF. If $f \geqq g$, then $f^+ \geqq g^+$ and $f^- \leqq g^-$. We obtain

$$\begin{aligned} \| Tf - Tg \|_1 &\leqq \| T_{(+)} f^+ - T_{(+)} g^+ \|_1 + \| T_{(-)} g^- - T_{(-)} f^- \|_1 \\ &\leqq \| f^+ - g^+ \|_1 + \| g^- - f^- \|_1 \\ &= \| f - g \|_1. \end{aligned}$$

The lemma implies that we can construct nonexpansive operators T in L_1, by defining order preserving nonexpansive operators T_1, T_2 in L_1^+ satisfying $T_1 0 = T_2 0 = 0$, and putting $T = T_1 \ominus T_2$. We remark that this lemma does not hold in L_p with $p > 1$. Simply let $\Omega = \{1, 2\}$, $\mu(\{1\}) = \mu(\{2\}) = \frac{1}{2}$. Identify the elements of L_p with vectors (f_1, f_2). Put $T_1(f_1, f_2) = (f_1, f_2)$ and $T_2 = (f_1, f_2) = (f_2, f_1)$, and consider $f = (1, 0)$ and $g = (0, -1)$.

LEMMA 2.4. *If T is order preserving in L_p $(1 \leqq p \leqq \infty)$ and $T0 = 0$, and if $T_{(+)}$ and $T_{(-)}$ are norm-decreasing in L_p^+, then T is norm-decreasing in L_p.*

PROOF. We have $-T_{(-)} f^- \leqq Tf \leqq T_{(+)} f^+$. Hence

$$| Tf |^p \leqq \operatorname{Max}(| T_{(-)} f^- |^p, | T_{(+)} f^+ |^p)$$

and

$$\begin{aligned} \| Tf \|_p^p &\leqq \int (| T_{(+)} f^+ |^p + | T_{(-)} f^- |^p) d\mu \\ &= \| T_{(+)} f^+ \|_p^p + \| T_{(-)} f^- \|_p^p \end{aligned}$$

$$\leqq \|f^+\|_p^p + \|f^-\|_p^p$$

$$= \int (|f^+|^p + |f^-|^p) d\mu$$

$$= \|f\|_p^p. \qquad \blacksquare$$

It may not be simple to check directly whether an operator is nonexpansive in L_p. Then the following theorem, which follows from a very general result of Browder [Br], can be helpful:

THEOREM 2.5. *If T is nonexpansive in L_1 and in L_∞, then T is nonexpansive in L_p for $1 < p < \infty$. Similarly, if T is nonexpansive in L_1^+ and in L_∞^+, then T is nonexpansive in L_p^+ for $1 < p < \infty$.*

For our purposes, the special case of this theorem, in which T is order preserving, will be sufficient. A simple argument for this case will be sketched in the appendix of this paper. The argument will also establish the following result, which may be new:

PROPOSITION 2.6. *If T is nonexpansive in L_1^+ and order preserving, and if T is norm-decreasing in L_∞^+, then T is norm-decreasing in L_p^+ $(1 \leqq p \leqq \infty)$.*

We shall see that it is not sufficient to assume T norm decreasing in L_1^+ and in L_∞^+ in this proposition.

If T is an order preserving nonexpansive operator in L_1^+, the domain of definition of T may be extended to the space of all nonnegative measurable functions. Let $0 \leqq f_1 \leqq f_2 \leqq \cdots$ be an increasing sequence of integrable functions tending to ∞ a.e., and put $Tf = \lim_{n\to\infty} T(f \wedge f_n)$. Tf can assume the value ∞. It is not difficult to check that the definition of T is independent of the specific choice of the sequence (f_n) and that T remains order preserving. Also $\|Tf - Tg\|_1 \leqq \|f - g\|_1$ remains valid for nonintegrable f, g with integrable difference. Clearly, if T deceases the L_∞^+-norm in the original range of definition, it does so in the larger domain. If T is defined in L_1 to begin with, the range of the extension is the set of all measurable functions f with $f^- \in L_1$. We can then further extend the range of definition of T, taking a sequence $0 = g_1 \geqq g_2 \geqq \cdots$ of integrable functions with $g_n \to -\infty$ a.e., and putting $Th = \lim T(h \vee g_n)$.

In particular, if T is order preserving and nonexpansive in L_1, and if T contracts the L_∞-norm then $T^k f$ is well defined for all $f \in L_p$ with $1 < p < \infty$ and $T^k f \in L_p$. (We have $T^k(-f^-) \leqq T^k f \leqq T^k f^+$ for all k and Proposition 2.6 yields $\|T^k f^+\|_p \leqq \|f^+\|_p$ and $\|T^k(-f^-)\|_p \leqq \|f^-\|_p$.)

3. Convergence in distribution

We now want to derive a theorem on convergence in distribution for the sequence $T^n f$. To attain full generality, we must first define convergence in distribution in σ-finite measure spaces.

It will be convenient to use the common notation I_t for the interval $\{s \in R: s > t\}$, when $t > 0$, and for $\{s \in R: s < t\}$, when $t < 0$. Let $\boldsymbol{D}$ be the family of measures γ on R with $\gamma(I_t) < \infty$ for all $t \neq 0$. $\gamma(\{0\}) = \infty$ is permitted. We shall mainly be interested in measures of the form $\gamma(B) = \mu(\{f \in B\})$, where f is an integrable function on a possibly infinite measure space (Ω, Σ, μ). γ is called the distribution of f. Observe that the sequence $\gamma_n(I_t)$ is bounded for any $t \neq 0$, if γ_n is the distribution of f_n and the sequence (f_n) is bounded in L_1.

Let $C_{b,0}$ denote the set of all bounded continuous functions on R which vanish in a neighborhood of 0. We say that the sequence $\gamma_n \in \boldsymbol{D}$ converges in distribution if $\gamma_n(R)$ converges and $\int h d\gamma_n$ converges to a finite limit for $h \in C_{b,0}$. The numbers $\gamma_n(R)$ or their limit may be infinite.

If γ_n is the distribution of f_n, $\gamma_n(R) = \mu(\Omega)$ and $\int h d\gamma_n = \int h \circ f_n d\mu$. Thus, we may say that f_n converges in distribution if $\int h \circ f_n d\mu$ converges for all $h \in C_{b,0}$.

Recall that, in the classical definition, the sequence $\gamma_n(R)$ is bounded and the convergence of $\int h d\gamma_n$ is requested for *all* bounded continuous h. For $\gamma \in \boldsymbol{D}$, the integrals $\int h d\gamma$ need not be well defined for all bounded continuous h. (There are also other reasons why $C_{b,0}$ seems more appropriate in the infinite case.) Anyway, the following characterization of convergence in distribution shows that the new definition is equivalent to the old one in the classical case.

THEOREM 3.1. *Let γ_n be a sequence of elements of $\boldsymbol{D}$ for which $\gamma_n(R)$ converges. Then the following assertions are equivalent*:

(i) *γ_n converges in distribution.*

(ii) *There exists a dense subset $D \subset R$ such that*

(a) *$F(t) = \lim_{n\to\infty} \gamma_n(I_t)$ exists for $t \in D$, and*

(b) *$F(t) \to 0$ for $t \to \pm\infty$.*

(iii) *Condition* (iia) *holds, and the family of measures γ_n is* tight; *i.e. for any $\varepsilon > 0$ there exists $t(\varepsilon) > 0$ with $\gamma_n(I_{t(\varepsilon)}) < \varepsilon$ and $\gamma_n(I_{-t(\varepsilon)}) < \varepsilon$ for all n.*

PROOF. (i) $\Rightarrow$ (ii). For $0 < t < s$, let $h_{t,s}$ be the function which equals 0 on $(-\infty, t]$, equals 1 on $[s, \infty)$ and is linear in between: $h_{t,s}(x) = (x-t)/(s-t)$ for $t \leqq x \leqq s$. $\int h_{t,s} d\gamma_n$ converges to a limit $F(t,s)$ with $\limsup \gamma_n(I_s) \leqq F(t,s) \leqq \liminf \gamma_n(I_t)$. Put $F(t) = \lim_{s\to t} F(t,s)$. It is simple to see that F is decreasing. Let $D \cap R^+$ be the set of positive continuity points of F. The construction of $D \cap R^-$ is symmetric. Routine arguments show (iia).

Now assume $\lim_{t\to\infty} F(t) = \alpha > 0$. Let $\varepsilon = \alpha/20$. There exists an n_1 and $s_1 > 0$ such that $n \geqq n_1$ implies $|\gamma_n(I_{s_1}) - \alpha| < \varepsilon$. Next find $t_1 > s_1$ and a continuous function h_1 with $0 \leqq h_1 \leqq 1$ and support in (s_1, t_1) and $|\int h_1 d\gamma_{n_1} - \alpha| < 2\varepsilon$. We can assume $\gamma_{n_1}(I_{t_1}) < \varepsilon$. Next find $s_2 > t_1$ and $n_2 > n_1$ such that $|\gamma_{n_2}(I_{s_2}) - \alpha| < \varepsilon$, etc. The continuation of the inductive construction yields sequences $s_1 < t_1 < s_2 < t_2 < \cdots$ and $n_1 < n_2 < n_3 < \cdots$ with

$$|\gamma_{n_i}(I_{s_i}) - \alpha| < \varepsilon, \qquad \gamma_{n_i}(I_{t_i}) < \varepsilon.$$

Moreover, we obtain continuous functions h_i with $0 \leqq h_i \leqq 1$ with support in (s_i, t_i) and

$$\left| \int h_i d\gamma_{n_i} - \alpha \right| < 2\varepsilon.$$

Put $h = h_1 + h_3 + h_5 + \cdots$. It is then not hard to show that $|\int h d\gamma_{n_i} - \alpha| < 4\varepsilon$ for odd i, and $\int h d\gamma_{n_i} < 4\varepsilon$ for even i, contradicting the convergence of $\int h d\gamma_n$. Hence $\lim_{t\to\infty} F(t) = 0$. The argument for $\lim_{t\to-\infty} F(t) = 0$ is symmetric.

(ii) $\Rightarrow$ (iii). Find $r(\varepsilon) > 0$ and $s(\varepsilon) < 0$ in D with $F(r(\varepsilon)), F(s(\varepsilon)) < \varepsilon/2$. Then find $n(\varepsilon)$ such that $n \geqq n(\varepsilon)$ implies $\gamma_n(I_{r(\varepsilon)}) < \varepsilon$ and $\gamma_n(I_{s(\varepsilon)}) < \varepsilon$. For large enough $t(\varepsilon) \geqq \mathrm{Max}(r(\varepsilon), -s(\varepsilon))$ the desired inequalities hold for the finitely many $n < n(\varepsilon)$.

(iii) $\Rightarrow$ (i). Let h be a bounded continuous function with support in the half-line $A = [a, \infty)$ where $a > 0$. We can assume that a is a continuity point of F and that $\gamma_n(\{a\}) = 0$ holds for all n. It is then simple to see that $\gamma_n(A) \to F(a)$. Let γ'_n denote the measure with $\gamma'_n(B) = \gamma_n(A \cap B)$. Then $\gamma'_n(R) \to F(a) < \infty$. We have $\gamma'_n(I_t) = \gamma_n(I_t)$ for $t \geqq a$ and $\gamma'_n(I_t) = \gamma_n(I_a)$ for $0 < t \leqq a$. The classical theorems on convergence in distribution for a bounded sequence of measures now imply the convergence of $\int h d\gamma'_n = \int h d\gamma_n$. (See, e.g., [Bau]).

We remark that convergence in distribution is called weak convergence in [Bau] and in most probability texts. But we shall use the functional analytic concept of weak convergence below, and prefer to distinguish the two notions.

THEOREM 3.2. *Let f_n be a sequence of measurable functions in (Ω, Σ, μ) and assume that the integrals $\int (f_n - t)^+ d\mu$ and the integrals $\int (f_n + t)^- d\mu$ exist and converge to finite limits for all $t > 0$ as $n \to \infty$. Then the sequence (f_n) converges in distribution. (The same result holds if n ranges through any directed set).*

PROOF. Let γ_n denote the distribution of f_n. For $t > 0$ put $g_t(x) = (x - t)^+$. Then $\int (f_n - t)^+ d\mu = \int g_t d\gamma_n$. The functions $h_{t,s}$, defined for $0 < t < s$ in step (i) $\Rightarrow$ (ii) of the previous proof, may be written as $h_{t,s} = (s - t)^{-1}(g_t - g_s)$. It follows that $\int h_{t,s} d\gamma_n$ converges to a finite limit $F(t,s)$. The argument in the previous proof establishes condition (iia).

The boundedness of the sequence $\int (f_n - 1)^+ d\mu$ implies

$$\gamma_n(I_t) = \mu(f_n > t) \leqq (t-1)^{-1} \operatorname{Sup}_n \int (f_n - 1)^+ d\mu \to 0 \qquad \text{for } t \to \infty.$$

$F(t) = \lim_{s \to t} F(t,s)$, and $F(t,s) \leqq \gamma_n(I_t)$ then yields $F(t) \to 0$ for $t \to \infty$. The argument for the negative half-line is symmetric. Hence (iib) holds and the proof is complete. ■

As an application, we obtain

THEOREM 3.3. *Let $T_1, T_2, \ldots$ be order preserving and nonexpansive operators in L_1 or in L_1^+, and assume that the operators T_k decrease the L_∞-norm. Then, for any $f \in L_1$ (resp. L_1^+), the sequence $f_n = T_n T_{n-1} \cdots T_1 f$ converges in distribution.*

PROOF. We have $f_{n+1} = T_{n+1} f_n$, and for $t > 0$, $T_{n+1}(f_n \wedge t) \leqq T_{n+1}(f_n^+ \wedge t) \leqq t$. Hence

$$\int (f_{n+1} - t)^+ d\mu \leqq \int (f_{n+1} - T_{n+1}(f_n \wedge t)) d\mu = \| T_{n+1} f_n - T_{n+1}(f_n \wedge t) \|_1$$

$$\leqq \| f_n - (f_n \wedge t) \|_1 = \int (f_n - t)^+ d\mu.$$

The symmetric argument yields $\int (f_{n+1} + t)^- d\mu \leqq \int (f_n + t)^- d\mu$. ■

REMARK. Assume that, for $0 \leqq t < s$, $T_{s,t}$ is order preserving and nonexpansive in L_1 (or L_1^+), and that $T_{s,t}$ decreases the L_∞-norm. Moreover, assume $T_{u,t} = T_{u,s} T_{s,t}$. Then $T_{u,s} f$ converges in distribution for $f \in L_1$. The proof is the same.

4. A nonlinear ergodic theorem

The nonlinear ergodic theorems of Baillon [Bai 1], [Bai 2] do not seem to apply to our class of operators, because our T need not be nonexpansive in any L_p with $1 < p < \infty$. Nevertheless, we shall obtain a nonlinear ergodic theorem in L_p $(1 < p < \infty)$ and in L_1 for finite μ. The idea is to look at the portion of $T^i f$ which lies between sufficiently close levels and to show a kind of approximate nonexpansiveness in L_2 for these portions.

THEOREM 4.1. *Let T be order preserving and nonexpansive in L_1, and assume that T decreases the L_∞-norm. For any $f \in L_p$ $(1 < p < \infty)$, the averages*

$$A_n f = n^{-1} \sum_{k=0}^{n-1} T^k f$$

converge weakly in L_p. *If* μ *is finite,* $A_n f$ *converges weakly in* L_1 *for* $f \in L_1$. *The same result holds for operators in* L_1^+.

(Recall that T is well defined in L_p.)

The proof will be given in a sequence of steps. In the first step, we show that it is enough to consider nonnegative f.

LEMMA 4.2. *Let* T *be an order preserving operator in a space* L_p, *where* $1 \leqq p < \infty$. *Assume that* $T_{(+)}$ *and* $T_{(-)}$ (defined in Section 2) *are norm-decreasing in* L_p^+. *If* $T_{(+)}^n f$ *and* $T_{(-)}^n f$ *converge weakly or strongly for all* $f \in L_p^+$ *then* $T^n g$ *converges weakly* (*resp. strongly*) *for all* $g \in L_p$. *The analogous assertion holds for the Cesàro-averages.*

PROOF. It follows from $(T^{n+1}g)^+ \leqq T(T^n g)^+$ that the sequence $\|(T^n g)^+\|_p$ is decreasing. Similarly $\|(T^n g)^-\|_p$ is a decreasing sequence. Let γ_+ and γ_- be the limits of these sequences. Given $\varepsilon > 0$, we can find N with

$$\|(T^N g)^+\|_p^p \leqq \gamma_+^p + \varepsilon \quad \text{and} \quad \|(T^N g)^-\|_p^p \leqq \gamma_-^p + \varepsilon.$$

For $n \geqq 1$, consider $g_1 = (T^{N+n}g)^+$ and $g_2 = T^n(T^N g)^+$. Then $g_1 \leqq g_2$ and $\|g_2\|_p^p \leqq \|g_1\|_p^p + \varepsilon$. The inequality $(g_2 - g_1)^p \leqq g_2^p - g_1^p$ yields $\|g_1 - g_2\|_p^p < \varepsilon$. Hence

$$\|(T^{n+N}g)^+ - T^n(T^N g)^+\|_p < \varepsilon^{1/p}.$$

Similarly

$$\|-(T^{n+N}g)^- + T^n(-(T^N g)^-)\|_p < \varepsilon^{1/p}.$$

Now assume that $T_{(+)}^n f$ and $T_{(-)}^n f$ converge weakly for $f \in L_p^+$. Then $\langle T^n(T^N g)^+, h\rangle$ and $\langle T^n(-(T^N g)^-), h\rangle$ are Cauchy sequences for $h \in L_q = L_p^*$ for $n \to \infty$. This shows that the sequence $\langle T^{n+N} g, h\rangle$ stays within distance $2\varepsilon^{1/p}$ of a Cauchy sequence. As $\varepsilon > 0$ was arbitrarily small $\langle T^n g, h\rangle$ must be a Cauchy sequence. Hence $T^n g$ converges weakly. The argument for the strong convergence is even simpler: the sequences $T^n(T^N g)^+$ and $T^n(-(T^N g)^-)$ are Cauchy sequences. For the corresponding assertion with the Cesàro averages just note that the first n terms in the averages do not matter for the limit. ■

It will also be sufficient to prove Theorem 4.1 for bounded $f \in L_p^+$. To see this, first observe that the boundedness of the sequence of L_p-norms of $T^k f$ and of $A_n f$ implies that the convergence of $\langle A_n f, h\rangle$ for all $h \in L_q$ follows when we prove it for a family of functions h which is dense in L_q. By the linearity of the scalar product we need only consider $h = 1_B$ with $\mu(B) < \infty$. If $f \in L_p^+$ is unbounded there exist bounded functions f' with $0 \leqq f' \leqq f$, for which $\|f' - f\|_1$ is arbitrarily small. We have $\|A_n f - A_n f'\|_1 \leqq \|f' - f\|_1$. ($f$ and f' need not be integrable — see section 2.) Hence

$$|\langle A_n f, 1_B\rangle - \langle A_n f', 1_B\rangle| \leqq \|f' - f\|_1.$$

If $\langle A_n f', 1_B\rangle$ converges for all f', $\langle A_n f, 1_B\rangle$ must converge.

Finally, we reduce the proof to the case $f \in L_1 \cap L_\infty^+$. Assume that Theorem 4.1 has been established in this case, and assume $f \in L_p \cap L_\infty^+$ is not integrable. Let $\varepsilon > 0$ and $h = 1_B$ with $\mu(B) < \infty$ be given. Let $\alpha > 0$ be a number with $\alpha\mu(B) < \varepsilon/4$. The sequence $\eta_k = \int (T^k f - \alpha)^+ d\mu$ is decreasing (by the proof of Theorem 3.3). We want to show that

$$\gamma(f) := \limsup \langle A_n f, h\rangle - \liminf \langle A_n f, h\rangle$$

is smaller than ε. As $\gamma(f) = \gamma(T^k f)$ holds for all k we can replace f by $T^k f$ in the proof, where k is so large that $|\eta_k - \mathrm{Inf}_j \eta_j| < \varepsilon/4$ holds. In other words, we can assume

$$|\eta_0 - \mathrm{Inf}_j \eta_j| < \varepsilon/4.$$

We define an operator S in L_1^+ by $Sg = (T(g + \alpha) - \alpha)^+$. S is order preserving, nonexpansive in L_1^+ and norm-decreasing in L_∞^+. We claim that

$$T^i(g + \alpha) \leqq S^i g + \alpha \qquad (i \geqq 0).$$

The case $i = 0$ is trivial, and

$$T^{i+1}(g + \alpha) \leqq T(S^i g + \alpha) \leqq (T(S^i g + \alpha) - \alpha)^+ + \alpha = S(S^i g) + \alpha.$$

Now take $g = (f - \alpha)^+$. Then $f \leqq g + \alpha$ implies $T^i f \leqq S^i g + \alpha$ for all $i \geqq 0$. Hence $(T^i f - \alpha)^+ \leqq S^i g$. We have $\int S^i g d\mu \leqq \eta_0 = \int g d\mu$ and $\int (T^i f - \alpha)^+ d\mu = \eta_i \geqq \eta_0 - \varepsilon/4$. Hence $\|S^i g - (T^i f - \alpha)^+\|_1 < \varepsilon/4$. This, together with $|(T^i f - \alpha)^+ - T^i f| \leqq \alpha$, yields

$$|\langle S^i g, h\rangle - \langle T^i f, h\rangle| < \alpha\mu(B) + \varepsilon/4 < \varepsilon/2.$$

Put $A_n(S) = n^{-1} \sum_{i=0}^{n-1} S^i$. We know that $\limsup \langle A_n(S) g, h\rangle = \liminf \langle A_n(S) g, h\rangle$ because we had assumed that the theorem had been established in the integrable case. Putting all this together, we obtain $\gamma(f) < \varepsilon$.

The next lemma is inspired by the beautiful paper of Djafari-Rouhani and Kakutani [DRK].

LEMMA 4.3. *Let (x_i) be a bounded sequence of vectors in a Hilbert space H. Assume that, for some $\varepsilon > 0$,*

$$\limsup_{i,j\to\infty} \operatorname{Sup}_k [\|x_{i+k} - x_{j+k}\|^2 - \|x_i - x_j\|^2] \leqq \varepsilon. \tag{4.1}$$

Put $a_n = n^{-1}\sum_{i=1}^n x_i$. If u, v are two weak limit points of the sequence (a_n), then

$$\|u - v\|^2 \leqq \varepsilon.$$

PROOF. There exist increasing sequences (m_k) and (n_k) with

$$\text{w-lim}\, a_{m_k} = u, \qquad \text{w-lim}\, a_{n_k} = v.$$

Taking subsequences, and applying the diagonal argument, we can assume that the following limits exist:

$$\lim m_k^{-1} \sum_{i=1}^{m_k} \|x_i\|^2 = \alpha,$$

$$\lim n_k^{-1} \sum_{i=1}^{n_k} \|x_i\|^2 = \beta,$$

$$\lim m_k^{-1} \sum_{i=1}^{m_k} \|x_i - x_j\|^2 = \varphi(j),$$

$$\lim n_k^{-1} \sum_{i=1}^{n_k} \|x_i - x_j\|^2 = \psi(j).$$

Again, taking subsequences, we can assume that

$$\lim m_k^{-1} \sum_{j=1}^{m_k} \varphi(j) = \varphi_\alpha \quad \text{and} \quad \lim n_k^{-1} \sum_{j=1}^{n_k} \varphi(j) = \varphi_\beta$$

exist, and that the same holds with all φ's replaced by ψ's. (4.1) implies that, for any $\varepsilon' > 0$, there exists $I(\varepsilon')$ such that $i, j \geqq I(\varepsilon')$ implies

$$\|x_{i+k} - x_{j+k}\|^2 \leqq \|x_i - x_j\|^2 + \varepsilon + \varepsilon'.$$

This in turn yields $\varphi(j+k) \leqq \varphi(j) + \varepsilon + \varepsilon'$ for sufficiently large j. We obtain $\limsup \varphi(j) \leqq \liminf \varphi(j) + \varepsilon$ and hence $|\varphi_\alpha - \varphi_\beta| \leqq \varepsilon$. Similarly, $|\psi_\alpha - \psi_\beta| \leqq \varepsilon$.

It follows from $\langle x_i, x_j\rangle = 2^{-1}(\|x_i\|^2 + \|x_j\|^2 - \|x_i - x_j\|^2)$ that

$$\langle a_m, a_n\rangle = \frac{1}{2m}\sum_{i=1}^m \|x_i\|^2 + \frac{1}{2n}\sum_{j=1}^n \|x_j\|^2 - \frac{1}{2mn}\sum_{i=1}^m\sum_{j=1}^n \|x_i - x_j\|^2. \tag{4.2}$$

If we put $m = m_k$ and let $k \to \infty$, then we obtain

$$\langle u, a_n \rangle = \frac{\alpha}{2} + \frac{1}{2n} \sum_{j=1}^{n} \| x_j \|^2 - \frac{1}{2n} \sum_{j=1}^{n} \varphi(j).$$

If we first put $n = m_k$, and then $n = n_k$ in this identity, we arrive at

$$\langle u, u \rangle = \alpha/2 + \alpha/2 - \varphi_\alpha/2$$

and

$$\langle u, v \rangle = \alpha/2 + \beta/2 - \varphi_\beta/2.$$

Similarly, putting $m = n_k$, and letting $k \to \infty$, and then putting $n = m_k$ and $n = n_k$ we obtain

$$\langle v, u \rangle = \beta/2 + \alpha/2 - \psi_\alpha/2$$

and

$$\langle v, v \rangle = \beta/2 + \beta/2 - \psi_\beta/2.$$

The last four identities yield $\| u - v \|^2 = \langle u - v, u - v \rangle = -\varphi_\alpha/2 + \varphi_\beta/2 - \psi_\beta/2 + \psi_\beta/2 \leqq \varepsilon.$ ∎

We now study the portion of $T^i f$ between two levels α and β.

LEMMA 4.4. *Let T be as in Theorem 4.1 and let $f \geqq 0$ be integrable. For $0 < \alpha < \beta$ put $f(i, \alpha, \beta) = (T^i f) \wedge \beta - (T^i f) \wedge \alpha$. For any $\eta > 0$ there exists $K \geqq 1$ such that $i, j \geqq K$, $k \geqq 0$, imply*

$$\| f(i+k, \alpha, \beta) - f(j+k, \alpha, \beta) \|_1 \leqq \| f(i, \alpha, \beta) - f(j, \alpha, \beta) \|_1 + \eta$$

and

$$\| f(i+k, \alpha, \beta) \wedge f(j+k, \alpha, \beta) \|_1 \geqq \| f(i, \alpha, \beta) \wedge f(j, \alpha, \beta) \|_1 - \eta.$$

PROOF. For any $t \geqq 0$ and $\gamma \geqq 0$ we have

$$\int (T^{\gamma+1} f - t)^+ d\mu \leqq \int (T^{\gamma+1} f - T(T^\gamma f \wedge t)) d\mu$$

$$\leqq \int (T^\gamma f - (T^\gamma f \wedge t)) d\mu = \int (T^\gamma f - t)^+ d\mu.$$

It follows that the sequences $\int (T^\gamma f - \alpha)^+ d\mu$, $\int (T^\gamma f - \beta)^+ d\mu$, and $\int T^\gamma f d\mu$ decrease, and, therefore, that the sequences $\int (T^\gamma f \wedge \alpha) d\mu$, $\int (T^\gamma f \wedge \beta) d\mu$ converge. Let Θ_α, Θ_β denote the limits. Suppressing i and j in the notation, we put (see Fig. 1)

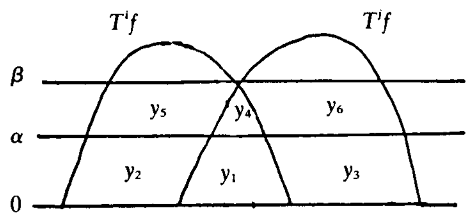

Fig. 1.

$$y_1 = (T^if) \wedge (T^jf) \wedge \alpha, \qquad y_4 = (T^if) \wedge (T^jf) \wedge \beta - y_1,$$

$$y_2 = (T^if) \wedge \alpha - y_1, \qquad y_5 = (T^if) \wedge \beta - (y_1 + y_2 + y_4),$$

$$y_3 = (T^jf) \wedge \alpha - y_1, \qquad y_6 = (T^jf) \wedge \beta - (y_1 + y_3 + y_4).$$

The functions y'_γ $(\gamma = 1,\dots,6)$ are defined in the same way except that T^if is replaced by $T^{i+k}f$ and T^jf by $T^{j+k}f$ (when k is fixed.) Observe that $y_1 + y_2 + y_3 = \mathrm{Max}(T^if, T^jf) \wedge \alpha$.

Let $\eta' > 0$ be much smaller than η. There exists K with

$$\int T^K f d\mu - \mathrm{Inf}_n \int T^n f d\mu < \eta'.$$

For $i \geqq K$, and for any g with $0 \leqq g \leqq T^if$, we have

$$0 < \int g d\mu - \int T^k g d\mu \leqq \int T^i f d\mu - \int T^{k+i} f d\mu < \eta'$$

for all $k \geqq 0$. In particular

$$\int T^k((T^if) \wedge \alpha) d\mu \approx \int ((T^if) \wedge \alpha) d\mu \approx \Theta_\alpha. \tag{4.3}$$

(We write $\approx$ when equality holds except for an error term the size of which is at most a multiple (independent of k) of η'. For functions the size will be the L_1-norm. We also write $\approx\leqq$ if $\leqq$ holds except for such an error term.)

Using $T^k((T^if) \wedge \alpha) \leqq (T^{i+k}f) \wedge \alpha$ and $\int (T^{i+k}f) \wedge \alpha d\mu \approx \Theta_\alpha$ we obtain

$$T^k((T^if) \wedge \alpha) \approx (T^{i+k}f) \wedge \alpha. \tag{4.4}$$

Similarly

$$T^k((T^if) \wedge \beta) \approx (T^{i+k}f) \wedge \beta, \tag{4.5}$$

and the same approximations are valid if i is replaced by j (K large enough for Θ_α and Θ_β to be approximated by the integrals, $i, j \geqq K$).

Put

$$u_\alpha = T^k(y_1 + y_2 + y_3) \quad \text{and} \quad u_\beta = T^k(y_1 + y_2 + \cdots + y_6) = T^k(\mathrm{Max}(T^i f, T^j f) \wedge \beta)$$

Using (4.5) we find $(T^{i+k}) \wedge \beta \approx \leqq u_\beta$. Similarly $(T^{j+k} f) \wedge \beta \approx \leqq u_\beta$. Hence

$$y_1' + \cdots + y_6' = \mathrm{Max}(T^{i+k} f, T^{j+k} f) \wedge \beta \approx \leqq u_\beta. \tag{4.6}$$

A similar argument yields

$$y_1' + y_2' + y_3' \approx \leqq u_\alpha. \tag{4.7}$$

Clearly $u_\alpha \leqq u_\beta$. We claim that

$$u_\alpha + y_4' + y_5' + y_6' \approx u_\beta. \tag{4.8}$$

As $u_\alpha \leqq \alpha$ and as the support of $(y_4' + y_5' + y_6')$ is contained in the set where $y_1' + y_2' + y_3' = \alpha$ holds, (4.8) follows from (4.6) and (4.7). In turn, (4.8) yields

$$\| y_4' + y_5' + y_6' \|_1 \approx \leqq \| u_\beta - u_\alpha \|_1 \leqq \| y_4 + y_5 + y_6 \|_1. \tag{4.9}$$

The rest of the proof is easy. (4.3) and $\int (T^{i+k} f) \wedge \alpha d\mu \approx \Theta_\alpha$ yield $\| y_1 + y_2 \|_1 \approx \| y_1' + y_2' \|_1$. The same relations with β instead of α give $\| y_1 + y_2 + y_4 + y_5 \|_1 \approx \| y_1' + y_2' + y_4' + y_5' \|_1$. Hence $\| y_4 + y_5 \|_1 \approx \| y_4' + y_5' \|_1$. By symmetry $\| y_4 + y_6 \|_1 \approx \| y_4' + y_6' \|_1$. Together with (4.9), these approximations yield $\| y_4 \|_1 \approx \leqq \| y_4' \|_1$. Using (4.9) once more we arrive at

$$\| y_5' + y_6' \|_1 \leqq \approx \| y_5 + y_6 \|_1.$$

Now, it remains to check that $\| y_5' + y_6' \|_1 = \| f(i+k, \alpha, \beta) - f(j+k, \alpha, \beta) \|_1$, $\| y_5 + y_6 \|_1 = \| f(i, \alpha, \beta) - f(j, \alpha, \beta) \|_1$, $\| y_4 \|_1 = \| f(i, \alpha, \beta) \wedge f(j, \alpha, \beta) \|_1$, and $\| y_4' \|_1 = \| f(i+k, \alpha, \beta) \wedge f(j+k, \alpha, \beta) \|_1$, and to make sure that η' is so small that all the multiples in the estimates are smaller than η.

The next lemma will be used to translate the L_1-estimates of the previous lemma into L_2-estimates.

Lemma 4.5. *Let (f_k) be a sequence of nonnegative measurable functions on (Ω, Σ, μ), bounded by $\gamma > 0$. Assume that $a_k = \mu(f_k > 0)$ tends to a finite limit a, that $b_k = \mu(f_k = \gamma)$ tends to b and that $\| f_k \|_2$ converges. Moreover, assume that, for any $\eta > 0$, there exists a number K such that $i, j \geqq K$, $k \geqq 0$ imply*

$$\| f_i \wedge f_j \|_1 \leqq \| f_{i+k} \wedge f_{j+k} \|_1 + \eta. \tag{4.10}$$

Then there exists, for any $\xi > 0$, *a number* K' *such that* $i, j \geqq K'$, $k \geqq 0$ *imply*

$$\|f_{i+k} - f_{j+k}\|_2^2 \leqq \|f_i - f_j\|_2^2 + \xi + 8(a-b)\gamma^2. \tag{4.11}$$

PROOF. Given $\xi > 0$ we can fix arbitrarily small numbers $\eta > 0$ and $\Delta > 0$, and find K such that (4.10) holds and the following inequalities hold for $K' \geqq K$ and ν, i, $j \geqq K'$:

$$|a_\nu - a| < \Delta, \qquad |b_\nu - b| < \Delta$$

and

$$|\, \|f_{i+k}\|_2^2 + \|f_{j+k}\|_2^2 - \|f_i\|_2^2 - \|f_j\|_2^2 \,| < \xi/4. \tag{4.12}$$

Put $\delta_i = a_i - b_i$ and $d(i,j) = \mu(f_i = \gamma \text{ and } f_j = \gamma)$. Then

$$\gamma d(i,j) \leqq \|f_i \wedge f_j\|_1 \leqq \gamma d(i,j) + \gamma(\delta_i + \delta_j).$$

Hence,

$$\begin{aligned} \gamma d(i+k, j+k) &\geqq \|f_{j+k} \wedge f_{i+k}\|_1 - \gamma(\delta_{i+k} + \delta_{j+k}) \\ &\geqq \|f_j \wedge f_i\|_1 - \gamma(\delta_{i+k} + \delta_{j+k}) - \eta \\ &\geqq \gamma d(i,j) - 2\gamma(a-b) - 2\gamma\Delta - \eta. \end{aligned}$$

We also have

$$\gamma^2 d(i,j) \leqq \int f_i f_j d\mu \leqq \gamma^2 d(i,j) + \gamma^2(\delta_i + \delta_j).$$

We obtain

$$\begin{aligned} \int f_{i+k} f_{j+k} d\mu &\geqq \gamma^2 d(i+k, j+k) \\ &\geqq \gamma^2 d(i,j) - 2\gamma^2(a-b) - 2\gamma^2\Delta - \gamma\eta \\ &\geqq \int f_i f_j d\mu - 4\gamma^2(a-b) - 4\gamma^2\Delta - \gamma\eta. \end{aligned}$$

Using (4.12), this yields

$$\begin{aligned} \|f_{i+k} - f_{j+k}\|_2^2 &= \|f_{i+k}\|_2^2 + \|f_{j+k}\|_2^2 - 2\int f_{i+k} f_{j+k} d\mu \\ &\leqq \|f_i - f_j\|_2^2 + \xi/4 + 8(a-b)\gamma^2 + 8\gamma^2\Delta + 2\gamma\eta. \end{aligned}$$

If Δ and η are so small that $2\gamma\eta < \xi/4$ and $8\gamma^2\Delta < \xi/4$ hold, (4.11) results. ∎

PROOF OF THEOREM 4.1. As mentioned before, we can assume that f is nonnegative, integrable, and bounded. Any bound $C>0$ for f is also a bound for all T^if. Theorem 3.3, with $T_i = T$, implies the convergence in distribution of the sequence T^if. Put

$$F(t)=\lim_{i\to\infty}\operatorname{Sup}\mu(T^if \geqq t).$$

Theorem 3.1 implies that $\lim_{i\to\infty}\mu(T^if\geqq t)=\lim_{i\to\infty}\mu(T^if>t)=F(t)$ holds for all t in the set D of positive continuity points of F, which is dense in $[0,C]$. Let $M=2M'$ be a large even integer, $\alpha_0=0$, $\alpha_{M+1}=C$, and let $0<\alpha_1<\alpha_2<\cdots<\alpha_M<C$ be elements of D. We assume $\alpha_{m+1}-\alpha_m\leqq 1$. Given any small $\chi>0$, we can assume

$$\sum_{m=0}^{M'}((\alpha_{2m+1}-\alpha_{2m})\|f\|_1)^{1/2}<\chi$$

and

$$F(\alpha_{2m})-F(\alpha_{2m-1})<\chi \qquad (m=1,\dots,M'). \tag{4.13}$$

(The big jumps of F must be placed between numbers α_{2m} and α_{2m+1}.) Now put

$$\begin{aligned} x_{0,i} &= (T^if)\wedge\alpha_1,\\ x_{m,i} &= (T^if)\wedge\alpha_{m+1}-(T^if)\wedge\alpha_m \qquad (m=1,\dots,M-1),\\ x_{M,i} &= ((T^if)-\alpha_M)^+ = T^if-(T^if)\wedge\alpha_M. \end{aligned}$$

Then $T^if=\sum_{m=0}^M x_{m,i}$. It follows from $0\leqq x_{m,i}\leqq(\alpha_{m+1}-\alpha_m)\leqq 1$ that

$$\int x_{m,i}^2 d\mu\leqq(\alpha_{m+1}-\alpha_m)\|x_{m,i}\|_1\leqq(\alpha_{m+1}-\alpha_m)\|f\|_1.$$

Hence

$$\sum_{m=0}^{M'}\|x_{2m,i}\|_2<\chi. \tag{4.14}$$

Now put $\alpha_{n,m}=n^{-1}\sum_{i=0}^{n-1}x_{m,i}$. Then $A_nf=\sum_{m=0}^M\alpha_{nm}$. Assume that the sequence A_nf does not converge weakly in L_2.

Then there exists an increasing sequence $1\leqq n_1<n_2<\cdots$ such that $A_{n_{2\gamma}}f$ converges weakly to a limit u, while $A_{n_{2\gamma+1}}f$ converges to a different weak limit v. Taking subsequences we can assume that the weak limits w-lim $\alpha_{n_{2\gamma},m}=:u_m$ and w-lim $\alpha_{n_{2\gamma+1},m}=v_m$ exist. Clearly $u=\sum_{m=0}^M u_m$ and $v=\sum_{m=0}^M v_m$. (4.14) yields

(4.15) $$\sum_{m=0}^{M'} \| u_{2m} - v_{2m} \|_2 < 2\chi.$$

If we put $\alpha = \alpha_{2m+1}$ and $\beta = \alpha_{2m+2}$ in Lemma 4.4, then $f(i, \alpha, \beta)$ is the element $x_{2m+1,i}$. The convergence in distribution of $T^i f$ also implies the convergence in distribution of $x_{2m+1,i}$ for $i \to \infty$. As $\mu(x_{2m+1,i} > 0)$ is a bounded sequence and $x_{2m+1,i} \leqq \alpha_{2m+2}$ is uniformly bounded, the sequence $\| x_{2m+1,i} \|_2$ converges. It follows that we can apply Lemma 4.5 to $f_i = x_{2m+1,i}$, $\gamma = \alpha_{2m+2} - \alpha_{2m+1}$, $b_k = \mu(T^k f \geqq \alpha_{2m+2})$ and $a_k = \mu(T^k f > \alpha^{2m+1})$. Then $b = F(\alpha_{2m+2})$, $a = F(\alpha_{2m+1})$, $\gamma = \alpha_{2m+2} - \alpha_{2m+1}$, and $a - b < \chi$. As $\xi > 0$ was arbitrarily small, (4.11) yields the assertion corresponding to (4.1) with $\|\cdot\| := \|\cdot\|_2$, $x_\nu := x_{2m+1,\nu}$ and with $\varepsilon := 8\chi(\alpha_{2m+2} - \alpha_{2m+1})^2$. Lemma 4.3 now implies that any two weak limit points of the sequence of averages $a_{n,2m+1}$ have a distance at most $\varepsilon^{1/2}$. We have proved

$$\| u_{2m+1} - v_{2m+1} \|_2 \leqq (8\chi)^{1/2}(\alpha_{2m+2} - \alpha_{2m+1}).$$

Using $\sum_{m=0}^{M} (\alpha_{m+1} - \alpha_m) \leqq C$ and (4.15) we find

$$\| u - v \|_2 \leqq \sum_{m=0}^{M} \| u_m - v_m \|_2 \leqq 2\chi + (8\chi)^{1/2} C.$$

As χ was arbitrarily small, $u = v$, and the proof is complete. ■

EXAMPLE. We now show by an example that an order preserving nonexpansive operator in L_1^+ which decreases the L_∞-norm need not be nonexpansive in L_p^+ for any p with $1 < p < \infty$.

We begin with a construction which is more general than needed here. Let (Ω, Σ, μ) be a measure space, and let $\{\tau_t, t \geqq 0\}$ be a family of invertible measure preserving transformations τ_t in Ω, such that the map

$$\tilde{\omega} = (\omega, t) \to \tilde{\tau}\tilde{\omega} = (\tau_t \omega, t)$$

of $\tilde{\Omega} = \Omega \times R^+$ into $\tilde{\Omega}$ and its inverse are measurable with respect to the product-σ-algebra $\Sigma \otimes B$, where B is the Borel-σ-algebra in R. Let λ denote the Lebesgue measure in R^+ and $\tilde{\mu} = \mu \otimes \lambda$ the product measure of μ and λ. $\tilde{\tau}$ is measure preserving in $\tilde{\Omega}$. For any $f \in L_1^+$ let

$$F = \{(\omega, t) \in \tilde{\Omega}\colon f(\omega) \geqq t\} \quad \text{and} \quad \tilde{\tau}F = \{(\omega, t)\colon (\tau_t^{-1}\omega, t) \in F\}.$$

Put $f'(\omega) = \lambda(t\colon (\omega, t) \in \tilde{\tau}F)$. It is not difficult to check that the operator $f \to Tf = f'$ is order preserving, nonexpansive in L_1^+ (by Lemma 2.2), and that T decreases the L_∞-norm.

Now consider the case where (Ω, Σ, μ) is the unit interval $[0, 1[$ with Lebesgue measure and where τ_t is the translation by t mod 1. Consider $g(\omega) = 1 - \omega$ and $f(\omega) = (g(\omega) - \varepsilon)^+$, where $\varepsilon > 0$ is a small number. Then Tg is the function g' with $g'(\omega) = \omega$, and Tf the function f' which is equal to g' in $[0, 1 - \varepsilon]$ and 0 in $]1 - \varepsilon, 1[$. It is simple to see that, for $1 < p < \infty$,

$$\|g - f\|_p^p \leqq \varepsilon^p \quad \text{and} \quad \|g' - f'\|_p^p \geqq \varepsilon(1 - \varepsilon)^p.$$

This shows that T is not nonexpansive in L_p^+.

T is *disjointly additive*, i.e., $T(f + g) = Tf + Tg$ holds for functions $f, g \in L_1^+$ with $f \wedge g = 0$. Roughly speaking, disjointly additive operators have the property (1) stated in the introduction, but not property (2).

We remark that the construction of T can be vastly generalized. It is not important to use a deterministic motion in the various levels. If one works with suitable stochastic or substochastic kernels, the description of T gets somewhat more complicated. On the other hand, this generalization yields a general representation theorem for order preserving disjointly additive nonexpansive operators in L_1^+. As this is independent of the present results, it will be given in a subsequent paper [KL]. An application of these operators in the case of finite state spaces has recently been given in [AK].

5. Speed limit operators

We now introduce a class of order preserving operators in the space L_1^+ of nonnegative Lebesgue integrable functions on the line $\Omega = R$ which can be used to show that strong convergence need not hold in Theorem 4.1, even if T is nonexpansive in L_p^+ for $1 \leqq p \leqq \infty$. Usually, an example of Genel and Lindenstrauss [GL] is used to show that strong convergence need not hold in Baillon's ergodic theorem. However, their example has been constructed for a different purpose. Therefore, it is unnecessarily complicated. Moreover, it depends on a fairly deep theorem of Kirszbraun [Ki], and additional work is needed to apply it to ergodic averages. Finally, an example in Hilbert space can not satisfy all of the present conditions (like the preservation of the order).

The idea of the construction below admits a number of generalizations. We remark that an infinite dimensional refinement has been used [Kr 2] to show that the condition of linearity cannot be dropped in the classical *pointwise* ergodic theorem of Hopf and Dunford–Schwartz.

Speed limit operators are determined by a decreasing nonnegative function φ on $\Omega = R$. φ need not be strictly decreasing. Heuristically we consider the upper half plane $\{(x, y): x \in R,\ y \geqq 0\}$ as a multi-lane road. The cars can only move to the right, keeping their y-coordinate. If (x, y) is the coordinate of a car at time t, then $\varphi(x)$ is the maximal permitted speed at time t for this car. If f is a nonnegative function, the set $G = \{(x, y): 0 \leqq y \leqq f(x)\}$ represents a cluster of cars. The car at (x, y) will drive at speed $\varphi(x)$ except if there is some $x' > x$ such that the "lane" with coordinate y is filled with cars in the interval $[x, x']$, and the car (x', y) has a smaller speed limit $\varphi(x')$ at this moment; then, none of the cars in this interval can go faster than $\varphi(x')$. The cluster G is transmitted into a cluster G_1 in one time unit. There is a function f_1 with $G_1 = \{(x, y): 0 \leqq y \leqq f_1(x)\}$. We put $Tf = f_1$. It will be convenient to formalize this only for a dense class of functions and to define T on L_1^+ by continuity.

Let F denote the class of nonnegative functions f of the form

$$f = \alpha \sum_{i=1}^{k} I(a_i, b_i)$$

where the $I(a_i, b_i)$ are indicator functions of intervals $[a_i, b_i[$ with $a_1 < b_1 \leqq a_2 < b_2 \leqq a_3 < \cdots < b_k$. Define c_i^t as the solution of

$$c_i^t = \int_0^t \varphi(b_i + c_i^s) ds$$

and

$$d_i^t = \operatorname{Min}(c_i^t, c_{i+1}^t + (a_{i+1} - b_i), c_{i+2}^t + (a_{i+1} - b_i) + (a_{i+2} - b_{i+1}), \ldots).$$

Equivalently the sequence d_i^t could be defined by backward induction:

$$d_k^t = c_k^t \quad \text{and} \quad d_i^t = \operatorname{Min}(c_i^t, d_{i+1}^t + (a_{i+1} - b_i)) \qquad (i = k-1, k-2, \ldots, 1);$$

c_i^t is the distance the cars in the interval $[a_i, b_i[$ could cover in the time interval $[0, t]$, if there were no further intervals $[a_j, b_j[$ with $j > i$, d_i^t is the distance they can actually cover. We put

$$a_i^t = a_i + d_i^t, \qquad b_i^t = b_i + d_i^t$$

and

$$T_t f = \alpha \sum_{i=1}^{k} I(a_i^t, b_i^t).$$

It follows from $d_i^t \leqq d_{i+i}^t + (a_{i+1} - b_i)$ that $b_i^t \leqq a_{i+1}^t$ holds. Therefore the intervals $[a_i^t, b_i^t]$ again satisfy $a_1^t < b_1^t \leqq a_2^t < \cdots$. As φ is decreasing, the sequence c_i^t is decreasing for any t. If we have $b_i = a_{i+1}$ for some i, then $d_i^t = d_{i+1}^t$, and hence $b_i^t = a_{i+1}^t$. It follows that $T_t f$ is well defined. Clearly, T_t is integral preserving in F.

Let us check what happens, if for some j with $1 \leqq j \leqq k$, the j-th interval is deleted. In other words, consider

$$g = \alpha \sum_{i \neq j} I(a_i, b_i).$$

Clearly, the movement of the intervals to the right of $[a_j, b_j[$ is not affected. Let $\bar{d}'_i$ be the distance the cars in $[a_i, b_i[$ can cover in the new situation. Then

$$T_t g = \alpha \sum_{i \neq j} [\bar{a}'_i, \bar{b}'_i[\qquad \text{where } \bar{a}'_i = a_i + \bar{d}'_i \quad \text{and} \quad \bar{b}'_i = b_i + \bar{d}'_i.$$

It can happen that $\bar{d}'_{j-1}$ is larger than d'_{j-1}. But then we must have $\bar{d}'_{j-1} = d'_j + (a_j - b_{j-1})$ and $\bar{b}'_{j-1} \leqq b'_j$. It follows that $[\bar{a}'_{j-1}, \bar{b}'_{j-1}[$ must be contained in the union of $[a'_{j-1}, b'_{j-1}[$ and $[a'_j, b'_j[$. This argument can be continued with $j-2$, $j-3$, etc., until we come to an interval which moves in the same way within the new situation as in the old one. Then all intervals left of it do the same. It follows that $T_t g \leqq T_t f$. T_t is order preserving in F. We leave it to the reader to verify $T_{t+s} f = T_t T_s f$.

Let us now extend the range of definition of T_t. If f assumes possibly distinct values $\alpha_i \geqq 0$ on the intervals $[a_i, b_i[$ above, then f can be written in the form $f = \sum_{j=i}^{l} f_j$ where the f_j belong to F and $\{f_1 > 0\} \supset \{f_2 > 0\} \supset \cdots \supset \{f_l > 0\}$. This larger class of functions is denoted by F_1, and we put

$$T_t f = \sum_{j=i}^{l} T_t f_j. \tag{5.1}$$

The condition $\{f_j > 0\} \supset \{f_{j+1} > 0\}$ is important for this definition. Formula (5.1) need not hold for *all* families $(f_j) \subset F$ with sum f. It follows from the above monotonicity considerations that $\{T_t f_1 > 0\} \supset \{T_t f_2 > 0\} \supset \cdots \supset \{T_t f_l > 0\}$. This easily yields the semigroup properly $T_{t+s} = T_t T_s$ in F_1. It is also easy to check that T_t is order preserving and integral preserving in F_1. It follows that T_t is nonexpansive in F_1 with respect to the L_1-norm. As F_1 is dense in L_1^+, T_t may be extended to a nonexpansive order preserving operator T_t in L_1^+.

Finally, let us show that T_t is nonexpansive with respect to the L_∞-norm in F_1 and hence in L_1^+. In other words, we want to show: If Δ is a positive number and if f, g are two elements of F_1 with $|f - g| \leqq \Delta$ then $|T_t f - T_t g| \leqq \Delta$. As in Lemma 2.1, we can assume $0 \leqq f \leqq g$. By an approximation argument, we can also assume that f and g have only rational values. We can even assume that f and g assume only integer values, because the operators T_t are positively homogeneous, i.e., they satisfy $T_t(\alpha f) = \alpha T_t f$. Without loss of generality, we have $\Delta = 1$. The general case then follows by considering a sequence $f = h_0 \leqq h_1 \leqq h_2 \leqq \cdots \leqq$

$h_n = g$ with $h_\gamma - h_{\gamma+1} \leqq 1$. Finally, we can assume $g = f + 1$ on $\{f > 0\}$ by enlarging g if necessary.

In the representation $f = \Sigma_{j=1}^{l} f_j$, we can assume that all f_j are indicator functions. We can take

$$f_1 = f \wedge 1, \qquad f_2 = (f \wedge 2) - f_1,$$

$$f_j = (f \wedge j) - \sum_{i=1}^{j-1} f_i, \qquad j = 2, \ldots, l.$$

l is the maximal value of f. Now let $g_1 = g \wedge 1$, $g_2 = (g \wedge 2) - g_1, \ldots, g_{l+1} = g - \Sigma_{i=1}^{l} g_i$ be the analogous layers for g. Then $f_j = g_{j+1}$ holds for $j \geqq 1$. Hence $T_t g = T_t f + T_t g_1$. Clearly $T_t g_1 \leqq 1$. Hence $T_t g - T_t f \leqq 1$. Theorem 2.5 now implies that T_t is nonexpansive with respect to all L_p-norms $(1 \leqq p \leqq \infty)$. We have proved:

THEOREM 5.1. *Let φ be any nonnegative decreasing function on $\Omega = R$. Then the construction above yields a semigroup $\{T_t, t \geqq 0\}$ of order preserving and integral preserving operators T_t in L_1^+ of the Lebesgue measure on R. The T_t satisfy $T_t 0 = 0$, they are positively homogeneous, and they are nonexpansive in L_p^+ for $1 \leqq p \leqq \infty$.*

(We remark that the operators T_t are linear only in the case where φ is constant.)

Now it will be easy to give the promised example.

EXAMPLE 5.2. φ will be piecewise constant. We shall construct a sequence $0 < t_1 < t_2 < t_3 < \cdots$ which is rapidly increasing and a sequence $1 > \beta_1 > \beta_2 > \cdots$ of positive numbers, rapidly decreasing to 0 and we put $\varphi(x) = 1$ for $x < t_1$, $\varphi(x) = \beta_k$ for $t_k \leqq x < t_{k+1}$ $(1 \leqq k < \infty)$. T will be the operator T_1 and f the indicator function of the unit interval.

If t_1 is larger than some integer n_1, then $T^k f$ is the translate of f for $k < n_1$. If n_1 is large $\| A_{n_1} f \|_2$ is small. Fix n_1 such that $\| A_{n_1} f \|_2 < \frac{1}{4}$ when $t_1 = n_1 + 1$. If β_1 is very small the translates $T^k f$ will be close to the indicator function of $[t_1 - 1, t_1[$ for many of the subsequent k's. we can therefore fix $\beta_1 > 0$ and n_2 with $\| A_{n_2} f \|_2 > \frac{3}{4}$. After some time, the support of $T^k f$ will lie to the right of t_1 and then the speed limit operator starts acting like a translation by β_1. If t_2 is sufficiently large we obtain the existence of an integer n_3 with $\| A_{n_3} f \|_2 < \frac{1}{4}$. The previous argument can now be repeated. If $\beta_2 < \beta_1$ is very small, arbitrarily many of the $T^k f$ will be close to $[t_2 - 1, t_2[$ in L_2-norm, and we can find n_4 with $\| A_{n_4} f \|_2 > \frac{3}{4}$. It is now clear that the sequences t_i and β_i can be determined in such a way that the sequence $\| A_n f \|_2$ diverges. Hence, $A_n f$ can not converge strongly.

REMARK. If one wants to extend the semigroup (T_t) to all of L_1, the most natural approach seems to be the following: First put $S_t f = T_t f^+ - T_t f^-$. Then consider the family P_t of all partitions $s = (s_i)$ where $0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = t$ and put

$$S_{(s)}f = S_{s_n - s_{n-1}} S_{s_{n-1} - s_{n-s}} \cdots S_{s_1 - s_0} f.$$

It is not difficult to show that

$$T_t f = \lim_s S_{(s)} f$$

exists (strongly in L_1), when s goes through P_t which is partially ordered by refinement. T_t is order preserving and nonexpansive in L_1, and $\{T_t, t \geqq 0\}$ is a semigroup. T_t is norm decreasing in L_∞, but need not be nonexpansive.

The trivial extension $Tf = Tf^+$ will preserve nonexpansiveness in L_p $(1 \leqq p \leqq \infty)$.

Appendix: On nonlinear interpolation

We now provide the proof of Proposition 2.6, a sketch of a proof of Theorem 2.5 for order preserving T, and the example announced in Section 2.

PROOF OF PROPOSITION 2.6. Recall that λ denotes the Lebesgue measure in R^+ and μ the measure in Ω. We define the potential energy of a measurable subset A of $\Omega \times R^+$ by

$$\mathrm{pot}(A) = \iint 1_A(w,h) h^{p-1} \lambda(dh) \mu(dw).$$

It $f \geqq 0$ is measurable and $F = \{(w,h) \colon 0 \leqq h \leqq f(w)\}$, then

$$\mathrm{pot}(F) = \int_\Omega \int_0^{f(\omega)} h^{p-1} \lambda(dh) \mu(dw) = \int_\Omega p^{-1} f^p d\mu = p^{-1} \|f\|_p^p.$$

If A and B are measurable subsets of $\Omega \times R$ with $\mu \otimes \lambda(B) \leqq \mu \otimes \lambda(A)$, and if there are numbers $0 < h_1 < h_2$ with $B \subset \{(w,h) \colon h \leqq h_2\}$ and $A \subset \{(w,h) \colon h_1 \leqq h\}$, then

$$\mathrm{pot}(B) \leqq h_2^{p-1} \mu \otimes \lambda(B) \leqq h_2^{p-1} \mu \otimes \lambda(A) \leqq (h_2/h_1)^{p-1} \mathrm{pot}(A).$$

Now let $\varepsilon > 0$ be small and $\eta > 0$ even much smaller, and consider the numbers $h_0 = 0$, $h_i = \varepsilon + (i-1)\eta$ $(i \geqq 1)$, and the functions $f_i = f \wedge h_i$, $g_i = Tf_i$. For $i \geqq 1$ put

$$F_i = \{(w, h): f_{i-1}(w) \leqq h < f_i(w)\} \quad \text{and} \quad G_i = \{(w, h): g_{i-1}(w) \leqq h < g_i(w)\}.$$

We have

$$\mu \otimes \lambda(G_i) = \| Tf_i - Tf_{i-1} \|_1 \leqq \| f_i - f_{i-1} \|_1 = \mu \otimes \lambda(F_i).$$

As T contracts the L_∞-norm, $f_i \leqq h_i$ implies $g_i \leqq h_i$. Hence G_i is contained in $\{(w, h): h \leqq h_i\}$, while F_i is contained in $\{(w, h): h \geqq h_{i-1}\}$.

If $\varepsilon > 0$ is small, $\mathrm{pot}(G_1) \leqq h_1^{p-1} \mu \otimes \lambda(G_1) \leqq \varepsilon^{p-1} \mu \otimes \lambda(F_1)$ is small. For $i \geqq 2$, we obtain $\mathrm{pot}(G_i) \leqq (h_i / h_{i-1})^{p-1} \mathrm{pot}(F_i) \leqq (((\varepsilon + \eta)/\varepsilon))^{p-1} \mathrm{pot}(F_i)$. Now $\| Tf \|_p \leqq \| f \|_p$ follows from

$$\| Tf \|_p^p = p \,\mathrm{pot}(\{w, h): 0 \leqq h < Tf\})$$

$$= p \sum_{i=1}^{\infty} \mathrm{pot}(G_i) \leqq p \,\mathrm{pot}(G_1) + p \sum_{i=2}^{\infty} ((\varepsilon + \eta)/\varepsilon)^{p-1} \mathrm{pot}(F_i)$$

and

$$p \sum_{i=1}^{\infty} \mathrm{pot}(F_i) = p \,\mathrm{pot}(F) = p \| f \|_p^p. \qquad \blacksquare$$

There is no need to prove the L_p-version of Proposition 2.6 because of Lemma 2.4.

The proof above can be modified to give a proof of Theorem 2.5 for order preserving T. We then have to prove $\| T\bar{f} - Tf \|_p \leqq \| \bar{f} - f \|_p$ for any $f, \bar{f} \in L_p$ with $\bar{f} \geqq f$. Now the level f (instead of 0) will be the 0-level of the potential in the copy of $\Omega \times R$, in which $f, \bar{f}$ are considered, and $g = Tf$ in the copy where $\bar{g} = T\bar{f}$ and g are considered. We look at $f_i = f + (h_i \wedge (\bar{f} - f))$, and at $g_i = Tf_i$ now. We leave the details to the reader. In this paper, we need Theorem 2.5 only in the order preserving case. The general case is a consequence of a general theorem of Browder [Br]. One can take $X_1 = L_1$, $X_2 = L_\infty$, $X = L_p$, $X_0 = L_\infty \cap L_1$ in that paper. The Riesz–Thorin theorem (see, e.g., Triebel [Tr], p. 135) implies that T is a linear interpolation system in the sense of Browder. If the operator T is only defined in L_1^+, one can extend it to the complex L_1 by putting $Tf := T((\mathrm{Re}\, f)^+)$ and again apply Browder's result. (We are indebted to A. Lenck for these remarks.)

EXAMPLE. It seems natural to inquire if, in Proposition 2.6, the assumption that T is nonexpansive in L_1 can be replaced by the weaker assumption that it is norm decreasing in L_1. We now construct an order preserving, disjointly additive operator T in L_1^+ (or L_1) which decreases the L_1-norm and the L_∞-norm, but which does not decrease the L_p-norm for any p with $1 < p < \infty$.

Take a space $\Omega = \{a, b, c\}$ consisting of three points with measure $\frac{2}{3}, \frac{2}{3}, 1$. As T shall be disjointly additive, we only have to define Tf when f has support in a single point.

We put

$$T(\alpha 1_{\{a\}}) = \tfrac{2}{3}(\alpha \wedge 1)^+ 1_{\{c\}}, \quad T(\beta 1_{\{b\}}) = \tfrac{4}{3}((\beta \wedge 2) - 1)^+ 1_{\{c\}}, \quad T(\gamma 1_{\{c\}}) = 0.$$

Heuristically, this means that all negative mass and the mass in c disappear. The positive mass sitting in a below level 1 is mapped to c, and the mass in a above that level disappears. The positive mass in b between level 1 and 2 is doubled and mapped to c, while all other mass in b disappears. It is simple to check that T decreases the L_1-norm and the L_∞-norm. Now consider the function $f = 1_{\{a\}} + 2 \cdot 1_{\{b\}}$. We have

$$\|f\|_p^p = \int f^p d\mu = \tfrac{2}{3} + 2^p \tfrac{2}{3} = \tfrac{2}{3}(1 + 2^p)$$

and

$$\|Tf\|_p^p + \int (Tf)^p d\mu = (\tfrac{2}{3} + \tfrac{4}{3})^p = 2^p.$$

It is simple to check that this is larger than $\frac{2}{3}(1 + 2^p)$ for $1 < p < \infty$.

REFERENCES

[AB] M. A. Akcoglu and D. Boivin, *Approximation of contractions by isometries*, in preparation.

[AK] M. A. Akcoglu and U. Krengel, *Nonlinear models of diffusion on a finite space*, preprint.

[Bai 1] J. B. Baillon, *Un théorème de type ergodique pour les contractions non-linéaires dans un espace de Hilbert*, Compt. Rend. Acad. Sci. Paris A, **280** (1975), 1511–1514.

[Bai 2] J. B. Baillon, *Comportment asymptotique des itérés de contractions non-linéaires dans les espaces L_p*, Compt. Rend. Acad. Sci. Paris, **286** (1978), 157–159.

[Bau] H. Bauer, *Probability Theory and Elements of Measure Theory*, Academic Press, New York, 1981.

[Br] F. E. Browder, *Remarks on nonlinear interpolation in Banach spaces*, J. Funct. Anal. **4** (1969), 390–403.

[DRK] B. Djafari-Rouhani and S. Kakutani, *Ergodic theorems for nonexpansive nonlinear operators in a Hilbert space*, preprint (1984).

[GL] A. Genel and J. Lindenstrauss, *An example concerning fixed points*, Isr. J. Math. **22** (1975), 81–86.

[Ki] M. D. Kirszbraun, *Uber die zusammenziehenden und Lipschitzschen Transformationen*, Fund. Math. **22** (1934), 77–108.

[Kr 1] U. Krengel, *Ergodic Theorems*, de Gruyter Studies in Math. **6**, Berlin–New York, 1985.

[Kr 2] U. Krengel, *An example concerning the nonlinear pointwise ergodic theorem*, Isr. J. Math. **58** (1987), 193–197 (this issue).

[KL] U. Krengel and M. Lin, *Disjointly additive order preserving operators in L_1*, in preparation.

[Tr] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.